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Journal of Life &Physical Sciences

*acta*SATECH 10 (1):102-108(2018)

A novel method for solving time dependent functional differential equations

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Abstract

In this paper, we apply the natural homotopy perturbation method (NHPM) for solving time dependent functional differential equations.The proposed technique which combines the Natural transform method (NTM) and homotopy perturbation method (HPM) reduces numerical computations and avoid round off errors. Direct application of the method, with no discretization or restrictive assumptions, leads to solutions which compare favorably with existing methods. Efficiency and reliability of the method are further demonstrated through several examples.

Keywords: Natural transform method; homotopy perturbation method; evolution equations; emden-fowler equations; cauchy reaction-diffusion equations; klein-gordon equations.

1. Introduction

Analytical techniques for nonlinear problems have been of considerable interest to scientists and engineers in recent years (Nadjfi & Ghorbani, 2009). Several methods exist in literature, useful in finding approximate solutions to nonlinear problems. Notable among these is homotopy perturbation method (HPM), first proposed by He (2008); which has successfully been applied to solve many types of linear and nonlinear functional equations.

Application of the method, which is a combination of standard homotopy and perturbation techniques, leads to analytic or approximate solutions for a wide variety of problems.

However, homotopy perturbation method has its limitations as the assumption of existence of small parameter, which forms the basis of almost all perturbation methods, limits its application, since majority of nonlinear problems have no small parameters (Mohyud Din & Noor, 2009).

The Laplace transform is totally incapable of handling nonlinear equations because of the difficulties caused by the nonlinear term (Gupta et al., 2013).

Attempt to deal with these nonlinearities, has led to different methods, among which is the homotopy perturbation transform method (HPTM) which is a combined form of the Laplace transform method and the homotopy perturbation method (Khan & Wu, 2011).

The advantage of this method is its capability of combining two powerful methods for obtaining exact solution for nonlinear equations.

Babolian et al (2009) applied homotopy perturbation method (HPM) to solve time-dependent differential equations and Gupta et al (2013) applied homotopy perturbation transform method (HPTM) to solve time-dependent functional differential equations.

The Natural transform is similar to Laplace integral transform (Maitama et al., 2017).

The basic motivation of this paper is the enhancement of the application of the Natural transform method by coupling it with the homotopy perturbation method known as Natural homotopy perturbation method (NHPM).

Maitama et al (2017) applied NHPM to solve linear and nonlinear Schrodinger equations.

The proposed method, as demonstrated, has broad applicability to Evolution equations,

Emden-Fowler equations, Cauchy reaction-diffusion equations and Klein-Gordon equations. The NHPM gives solution in convergent series form with easily computable components, which compares favorably with existing methods. Thus NHPM is a powerful technique useful in the solution of time dependent functional differential equations.

2. Natural Homotopy Perturbation Method (NHPM)

We illustrate the basic idea of NHPM by considering a general nonlinear partial differential equation with the initial conditions of the form (Khan & Wu, 2011)

$$DV(x,t) + RV(x,t) + NV(x,t) = g(x,t)$$

$$V(x,0) = h(x,t), V_t(x,0) = f(x) \tag{2.1}$$

where D is the second order linear differential operator

$D = \frac{\partial^2}{\partial t^2}$, R is the linear differential operator of less order than D, N represents the general nonlinear differential operator and $g(x,t)$ is the source inhomogenous term.

Applying the Natural transform to equation (2.1) subject to the given initial condition, we have

$$N^+[DV(x,t)] + N^+[RV(x,t)] + N^+[NV(x,t)] = N^+[g(x,t)] \tag{2.2}$$

using property of Natural transform, we have

$$V(x,s,u) = \frac{1}{s}h(x) + \frac{u}{s^2} f(x) - \frac{u^2}{s^2} N^+[RV(x,t)] - \frac{u^2}{s^2} N^+[NV(x,t)] + \frac{u^2}{s^2} N^+[g(x,t)] \tag{2.3}$$

Taking the inverse Natural transform of equation (2.3), we have

$$V(x,t) = G(x,t) - N^{-1}[\frac{u^2}{s^2} N^+[RV(x,t) + NV(x,t)]] \tag{2.4}$$

where $G(x,t)$ is a term arising from the source term and the prescribed initial condition.

Now, applying the homotopy perturbation method (HPM)

$$V(x,t) \sum_{n=0}^{\infty} P^n V_n(x,t) \tag{2.5}$$

and the nonlinear term can be decomposed as

$$NV(x,t) = \sum_{n=0}^{\infty} P^n H_n(V) \tag{2.6}$$

where $H_n(v)$ can be evaluated using the following formula:

$$H_n(v_1, v_2, \dots, v_n) = \frac{1}{n!}\frac{\partial^n}{\partial P^n}\left[N \sum_{j=0}^{n} P^j V_j\right]_{p=0}; n = 0,1,2, \dots \tag{2.7}$$

Substituting equations (2.5) and (2.6) in (2.4); we have

$\sum_{n=0}^{\infty} P^n V_n(x,t) = G(x,t) -$
$P\left(N^{-1}\left[\frac{u^2}{s^2}N^+[R\sum_{n=0}^{\infty} P^n V_n(x,t) +\right.\right.$
$\left.\left.\sum_{n=0}^{\infty} P^n H_n(v)]\right]\right)$ (2.8)

which is the coupling of the Natural transform and the homotopy perturbation method using

He's polynomials.

Comparing the coefficient of like powers of *P*, we obtain the following approximation:

$$P^0 : V_0(x,t) = G(x,t)$$

$$P^1 : V_1(x,t), = -N^{-1}\left[\frac{u^2}{s^2}N^+[RV_0(x,t) + H_0(v)]\right]$$

$$P^2 : V_2(x,t) = -N^{-1}\left[\frac{u^2}{s^2}N^+[RV_1(x,t) + H_1(v)]\right]$$

$$P^3 : V_2(x,t) = -N^{-1}\left[\frac{u^2}{s^2}N^+[RV_2(x,t) + H_2(v)]\right] \quad (2.9)$$

and so on.

Thus, the series solution of equation (2.1) is

$$V(x,t) = \lim_{k\to\infty}\sum_{n=0}^{k} V_n(x,t)$$
(2.10)

3. Applications

Here, we apply the Natural Homotopy Perturbation Method (NHPM) to some time- dependent differential equations:

3.1 Evolution equation

Babolian et al (2009) and Gaji et al (2007) used HPM to solve the given problem using initial condition $V_0(x,t) = 0$ as initial guesses.

Example 3.1.1 Consider the equation

$y_t - y_{xxxx} = 0$ (3.1)

with initial condition $y(x,0) = \sin x$.

applying the natural homotopy perturbation method, we have

$$\sum_{n=0}^{\infty} p^n y_n(x,t) = \sin x + P\left(\left[N^{-1}\left[\frac{u}{s}N^+\left(\sum_{n=0}^{\infty} p^n y_{nxxxx}\right)\right]\right]\right) \quad (3.2)$$

Computing the coefficients of like powers of *P* in eqn (3.2); we obtain the following approximations.

$$P^0 : y_0(x,t) = \sin x,$$

$$P^1 : y_1(x,t) = N^{-1}\left[\frac{u}{s}N^+(y_{0xxxx})\right] = N^{-1}\left[\frac{u}{s}N^+(\sin x)\right] = t\sin x$$

$$P^2 : y_2(x,t) = N^{-1}\left[\frac{u}{s}N^+(y_{1xxxx})\right] = \frac{t^2}{2!}\sin x$$

$$P^3 : y_3(x,t) = N^{-1}\left[\frac{u}{s}N^+(y_{2xxxx})\right] = \frac{t^3}{3!}\sin x \quad (3.3)$$

and so on.

Thus, the series solution of equation (3.1) is given by

$$\begin{aligned} y(x,t) &= \lim_{k\to\infty}\sum_{n=0}^{k} y_n(x,t) \\ &= y_0(x,t) + y_1(x,t) + y_2(x,t) + y_3(x,t) + \cdots \\ y(x,t) &= \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots\right)\sin x = e^t \sin x \end{aligned}$$

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(3.4)

The exact solution is in close agreement with the result obtained by HPM (Babolian et al, 2009) and HPTM (Gupta et al, 2013).

3.2 Cauchy reaction-diffusion equation

Example 3.2.1: Consider the equation (Babolian et al, 2009)

$y_t = y_{xx} - y$ (3.5)

with initial condition $y(x,0) = e^{-x} + x$.

Applying the natural homotopy perturbation method, we have

$$\sum_{n=0}^{\infty} p^n y_n(x,t) = e^{-x} + x + P\left(N^{-1}\left[\frac{u}{s}N^{+}\left(\sum_{n=0}^{\infty} p^n y_{nxx} - \sum_{n=0}^{\infty} p^n y_n\right)\right]\right) \quad (3.6)$$

Computing the coefficients of like powers of P in eqn (3.6), we obtain the following approximations

$$P^0 : y_0(x,t) = e^{-x} + x$$

$$P^1 : y_1(x,t) = N^{-1}\left[\frac{u}{s}N^{+}(y_{0xx} - y_0)\right] = -xt$$

$$P^2 : y_2(x,t) = N^{-1}\left[\frac{u}{s}N^{+}(y_{1xx} - y_1)\right] = x\frac{t^2}{2!}$$

$$P^2 : y_3(x,t) = N^{-1}\left[\frac{u}{s}N^{+}(y_{2xx} - y_2)\right] = -x\frac{t^3}{3!}$$

$$P^3 : y_3(x,t) = N^{-1}\left[\frac{u}{s}N^{+}(y_{2xx} - y_2)\right] = -x\frac{t^3}{3!} \quad (3.7)$$

and so on.

Thus, the series solution of equation (3.5) is given by

$$\begin{aligned} y(x,t) &= \lim_{k\to\infty}\sum_{n=0}^{k} y_n(x,t) \\ &= y_0(x,t) + y_1(x,t) + y_2(x,t) + y_3(x,t) + \cdots \end{aligned}$$

and

$$y(x,t) = e^{-x} + x\left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \ . \ . \ .\right) = e^{-x} + xe^{-t} \quad (3.8)$$

The exact solution is in close agreement with the result obtain by HPM (Babolian et al, 2009) and HPTM (Gupta et al, 2013).

Example 3.2.2: Consider the equation (Babolian et al, 2009)

$$y_t = y_{xx} - (4x^2 - 2t + 2)y \quad (3.9)$$

with initial condition $y(x,0) = e^{x^2}$.

Applying the natural homotopy perturbation method, we have

$$\sum_{n=0}^{\infty} p^n y_n(x,t) = e^{x^2} + P\left(N^{-1}\left[\frac{u}{s}N^{+}\left(\sum_{n=0}^{\infty} p^n y_{nxx} - (4x^2 - 2t + 2)\sum_{n=0}^{\infty} p^n y_n\right)\right]\right) \quad (3.10)$$

Computing the coefficients of like powers of P in eqn (3.10), we obtain the following approximations

$$P^0 : y_0(x,t) = e^{x^2}$$

$$P^1 : y_1(x,t) = N^{-1}\left[\frac{u}{s}N^{+}\left(y_{0xx} - (4x^2 - 2t + 2)y_0\right)\right] = e^{x^2}t^2$$

$$P^2 : y_2(x,t) = N^{-1}\left[\frac{u}{s}N^{+}(y_{1xx} - (4x^2 - 2t + 2)y_1)\right] = e^{x^2}\frac{t^4}{2!}$$

$$P^3 : y_3(x,t) = N^{-1}\left[\frac{u}{s}N^{+}\left(y_{2xx} - (4x^2 - 2t + 2)y_2\right)y_2\right] = e^{x^2}\frac{t^6}{3!} \quad (3.11)$$

and so on.

Thus, the series solution of equation (3.9) is given by

$$y(x,t) = \lim_{k\to\infty}\sum_{n=0}^{k} y_n(x,t)$$

and

$$y(x,t) = e^{x^2}\left(1 + t^2 + \frac{t^4}{2!} + \frac{t^6}{3!} + \cdots\right) = e^{x^2+t^2} \quad (3.12)$$

The exact solution is in close agreement with the result obtain by HPM (Babolian et al, 2009), HPTM (Gupta et al, 2013) and HAM (Batanieh et al, 2008).

3.3 Emden-Fowler equation

Example 3.3.1: Consider the equation (Babolian et al, 2009)

$$y_{xx} + \frac{2}{x}y_x - (5+4x^2)y = y_t + (6-5x^2-4x^4) \quad (3.13)$$

with initial condition $y(x,0) = x^2 + e^{x^2}$.

Applying the natural homotopy perturbation method, we have

$$\sum_{n=0}^{\infty} P^n y_n(x,t) = x^2 + e^{x^2} + P\left(N^{-1}\left[\frac{u}{s}N^+\left(\sum_{n=0}^{\infty} P^n y_{nxx} + \frac{2}{x}\sum_{n=0}^{\infty} P^n y_{nx} -(5+4x^2)\sum_{n=0}^{\infty} P^n y_n - (6-5x^2-4x^4)\right)\right]\right) \quad (3.14)$$

Computing the coefficients of like powers of P in eqn (3.14) we obtain the following approximations

$$P^0 : y_0(x,t) = x^2 + e^{x^2}$$

$$P^1 : y_1(x,t) = N^{-1}\left[\frac{u}{s}N^+\left(y_{0xx} + \frac{2}{x}y_{0x} - (5+4x^2)y_0 - (6-5x^2-4x^4)\right)\right]$$

$$P^2 : y_2(x,t) = N^{-1}\left[\frac{u}{s}N^+\left(y_{1xx} + \frac{2}{x}y_{1x} - (5+4x^2)y_1\right)\right] = e^{x^2}\frac{t^2}{2!}$$

$$P^3 : y_3(x,t) = N^{-1}\left[\frac{u}{s}N^+\left(y_{2xx} + \frac{2}{x}y_{2x} - (5+43x^2)y_2\right)\right] = e^{x^2}\frac{t^3}{3!}$$

and so on. Thus, the series solution of equation (3.13) is given by

$$y(x,t) = \lim_{k\to\infty}\sum_{n=0}^{\infty} y_n(x,t)$$

and

$$y(x,t) = x^2 + e^{x^2}\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!}| + \cdots\right) = \quad (3.16)$$

The exact solution is in close agreement with the result obtained by HPM (Babolian et al, 2009), HPTM (Gupta et al, 2013) and HAM (Batanieh et al, 2008).

Example 3.3.2: Consider the equation (Babolian et al, 2009)

$$y_{xx} + \frac{2}{x}y_x - (5+4x^2)y = y_{tt} + (12x - 5x^3 - 4x^5) \quad (3.17)$$

with initial condition

$$y(x,0) = x^3 + e^{x^2};\ y_t(x,0) = -e^{x^2}.$$

Applying the natural homotopy perturbation method, we have

$$\sum_{n=0}^{\infty} P^n y_n(x,t) = x^3 + e^{x^2} - te^{x^2} + P\left(N^{-1}\left[\frac{u^2}{s^2}N^+\left(\sum P^n y_{nxx} + \frac{2}{x}\sum P^n y_{nx} -(5+4x^2)\sum P^n y_n - (12x-5x^3-4x^5)\right)\right]\right) \quad (3.18)$$

Computing the coefficients of like powers of P in eqn (3.18) we obtain the following approximations

$$P^0 : y_0(x,t) = x^3 + (1-t)e^{x^2}$$

$$P^1 : y_1(x,t) = N^{-1}\left[\frac{u^2}{s^2}N^+\left(y_{0xx} + \frac{2}{x}y_{0x} - (5+4x^2)y_0\right) - (12x-5x^3-4x^5)\right] e^{x^2}\left(\frac{t^2}{2!} - \frac{t^3}{3!}\right)$$

$$P^2 : y_2(x,t) = N^{-1}\left[\frac{u^2}{s^2}N^+\left(y_{1xx} + \frac{2}{x}y_{1x} - (5+4x^2)y_1\right)\right] = e^{x^2}\left(\frac{t^4}{4!} - \frac{t^5}{5!}\right)$$

$$P^3 : y_3(x,t) = N^{-1}\left[\frac{u^2}{s^2}N^+\left(y_{2xx} + \frac{2}{x}y_{2x} - (5+4x^2)y_2\right)\right] = e^{x^2}\left(\frac{t^6}{6!} - \frac{t^7}{7!}\right) \quad (3.19)$$

and so on.

Thus the series solution of equation (3.19) is given by

$$y(x,t)=\lim_{k\to\infty}\sum_{n=0}^{k}y_n(x,t)$$

and

$$\begin{aligned} y(x,t) &= x^3+e^{x^2}\left(1-t+\frac{t^2}{2!}-\frac{t^3}{3!}+\frac{t^4}{4!}-\frac{t^5}{5!}+\frac{t^6}{6!}-\frac{t^7}{7!}+\cdots\right) \\ &= x^3+e^{x^2-t} \end{aligned} \qquad (3.20)$$

The exact solution is in close agreement with the result obtained by HPM (Babolian et al, 2009), HPTM (Gupta et al, 2013) and HAM (Batanieh et al, 2008).

Klein-Gordon equation

Example 3.4.1: Consider the linear Klein-Gordon equation (Chowdhury & Hashini, 2009)

$$y_{tt} - y_{xx} = y \qquad (3.21)$$

with initial condition $y(x,0) = 1 + \sin x$, $y_t(x,0) = 0$.

Applying the natural homotopy perturbation method; we have

$$\sum_{n=0}^{\infty}p^n y_n(x,t)=1+\sin x+ P\left(N^{-1}\left[\frac{u^2}{s^2}N^+\left(\sum_{n=0}^{\infty}p^n y_n+\sum_{n=0}^{\infty}p^n y_{nxx}\right)\right]\right) \qquad (3.22)$$

Computing the coefficients of like powers of P in eqn (3.22), we obtain the following approximations

$$P^0 : y_0(x,t) = 1+\sin x$$

$$P^1 : y_1(x,t) = N^{-1}\left[\frac{u^2}{s^2}N^+(y_0+y_{0xx})\right] = \frac{t^2}{2!}$$

$$P^2 : y_2(x,t) = N^{-1}\left[\frac{u^2}{s^2}N^+(y_1+y_{1xx})\right] = \frac{t^4}{4!}$$

$$P^3 : y_3(x,t) = N^{-1}\left[\frac{u^2}{s^2}N^+(y_2+y_{2xx})\right] = \frac{t^6}{6!} \qquad (3.23)$$

and so on.

Thus the series solution of equation (3.21) is given by

$$y(x,t)=\lim_{k\to\infty}\sum_{n=0}^{k}y_n(x,t)$$

and

$$y(x,t)=\sin x+\left(1+\frac{t^2}{2!}+\frac{t^4}{4!}+\frac{t^6}{6!}+\cdots\right)=\sin x+\cosh t \qquad (3.24)$$

The exact solution is in close agreement with the result obtained by HPM (Chowdhury & Hashini, 2009) and HPTM (Gupta et al, 2013).

4. Conclusion

In this paper, the natural homotopy perturbation method is successfully applied to functional equations.

The advantage of the method over the Adomian decomposition method is that it is not necessary to Compute Adomian polynomials to solve nonlinear problems.

The method reduces volume of numerical computation while maintaining high accuracy of numerical results and can be considered an improvement on existing techniques with wide applications in Science and Engineering.

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