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On the application of steady state transition probability in g/m/1/k queueing system

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Abstract

We consider a general arrival and Markovian service time queueing system with one server under first come first served discipline, where the ij element of transition probability is given as matrix F and the system can accommodate finite number of arrival, K. Transition probability matrix and steady states probabilities were obtained. Numerical illustration is considered on D/M/1/5 queue, where the customers arrive at a rate of one per unit time and for which the mean of the exponential service time equals to $\frac{3}{4}$ so as to reflect its usefulness in solving the real life problem. This gives the probability β_i, that i customers complete their service during the period k^{th} and $(k+1)^{th}$ arrivals for $\beta_i, i = 0, 1, 2, \cdots, 5$ as $\beta_0 = 0.263597, \beta_1 = 0.351463,$ $\beta_2 = 0.234309,$ $\beta_3 = 0.104137, \beta_4 = 0.034712,$ $\beta_5 = 0.009257$ and its corresponding stationary probability π, of an arrival finding i customers already present as $\pi = (0.473099, 0.258124, 0.14073, 0.076003, 0.038326, 0.013719)$.

Keywords: G/M/1/K Queueing System, Markovian arrival, Markovian service, Transition Probability

Introduction

The G/M/1/K queueing system is a system of general arrival and Markovian service with one server in which the system can accommodate a definite number of K arrivals while the (K+1) customers upward will be prevented from entry the system. When we examined the M/G/1/K queue we saw that the matrix of transition probabilities is of size $K \times K$, since the number of customers that a departing customer leaves behind has to lie between zero and $K - 1$ inclusive. In particular, a departing customer could not leave K customers behind. In the *G/M/*1/*K* queue, the transition probability matrix is of size $(K + 1) \times (K + 1)$ since an arrival can find any number of customers present between zero and K inclusive. Of course, when an arrival finds K customers present, that customer is not admitted to the system, but is lost. **Kleinrock** (1975) and Medhi (1980) helped in classification of queueing system into various types depending on the arrival and service times situations and number of available space in the queue. **Lucantoni** (1993), Ross 1997 and **Meini** (1997) investigated various simulation techniques in queueing system while **Bolch** (1998) discussed various Queueing Networks and Markov Chains applications. Stochastic application of queueing theory which involves various probability distributions and its application is discussed in **Law** (2000) and the new Convergence Results on Functional Techniques for the Numerical Solution of M/G/1 Type Markov Chains is established. **Agboola** (2007) studied a single server queue where the inter arrival time is Markovian time and service time is general. This model generalizes the well known M/G/1 queue. The waiting time process is directly analysed by solvin the lindley's equation using transform method. The Laplace Stieltjes transforms (LST) of the steady state waiting time and queue length distribution are both derived, and used to obtain recursive equations for the calculation of moments. k - server queue is discussed in **Agboola** (2010), where the inter arrival time is Markovian and service time is Markovian, General and Erlang distributed. This model generalizes the M/M/K, M/G/K and M/Er/K queues. The departure distribution is directly analysed by using probability generating function to derive the service time, waiting time and sojourn time distribution under single server with general service time. The recursive equation is then used to obtain blocking probability for the Markov inter arrival with K –server under general service time. G/M/1 and G/M/K are discussed in **William** (2009), where the service process has exponential distribution with mean service time $\frac{1}{\mu}$. i.e. $B(x) = 1 - \exp(-\mu x),\ x \geq 0$, while the arrival process is general with mean inter arrival time equal to $\frac{1}{\lambda}$. Customers arrive individually and their inter arrival times are independent and identically distributed. To represent this system by a Markovian, it is necessary to keep track of time that passes between arrivals since the distribution of inter arrival times does not in general possess the memoryless property of the exponential. As was the case for the M/G/Q queue, a two – component state descriptor may be used; the first to indicate the number of customers present and the second to indicate the elapsed time since the previous arrival. In this way, the G/M/1 queue can be solved using the method of supplementary variables. It is also possible to define a Markov chain embedded within the G/M/1 queue. The embedded time instants are precisely the instants of customer arrivals, since the elapsed inter arrival time at these moments is known as zero. This allows us to form a transition probability matrix and to compute the distribution of customers as seen by an arriving customer. **Charan (2012)** investigated the single server queueing system wherein the arrival of the units follow a Poisson process with varying arrival rates in different states. The server may take a vacation of a fixed duration or may continue to be available in the system for next service. Probability generating function of the units present in the system and various performance indices such as expected number of units in the queue and in the system, average waiting time, e.t.c. are obtained. **Michiel** (2017) analysed a non- classical discrete time queueing model where customers demand variable amounts of work from a server that is able to perform this work at a varying rate. The service demands of the customers are integer numbers of work units. They are assumed to be independent and identically distributed (i.i.d) random variables. The service capacities, i.e., the numbers of work units that the server can process in the consecutive slots, are also assumed to be independent and identically distributed and their common probability generating functions is assumed to be rational. New customers arrive in the queue system according to a general independent arrival process. An analysis method, which is based on complex contour integration, is presented and expressions are obtained for the probability generating functions, the mean values and the tail probabilities of the customer delay and the system steady state with numerical example illustration. **Charan (2019)** considered a single server queueing system with batch arrival of the units. The provision of optimal service after availing essential service is available in the system, and it is assumed that the server may break down during any stage of the service of the units and provides the repair facility immediately. The server may also avail the vacation under Bernoulli vacation policy after completion of the service of the units. The supplementary variable approach with probability generating function is applied to analyse the system to find the system performance quantity and numerical illustration is considered to obtain the system state probabilities and queueing reliability indices.

Nomenclature

$B(x)$: Service Process

$A(t)$: Arrival Distribution

$a(t)$: Arrival Probability Density Function

M_K: The number of Customers Present in a G/M/1 queue just Prior to the k^{th} Arrival

B_{k+1}: The number of Service Completions that occur between the arrival of the k^{th} customer and that of the $(k+1)^{th}$ customer

f_{ij}: The ij element of the transition probability matrix F

β_i: The probability that i customers complete their service during the period k^{th} and $(k+1)^{th}$ arrivals

π_i: Stationary probability of an arrival finding i customers already present

$G_B(z)$: Z-transform of the number of service completions that occur during an interarrival period

ξ: The root of the equation $G_B(z)$

$E[N_A]$: The mean number in this system at arrival epoch

λ: Arrival rate

W_q: Waiting time distribution function of the customer in the queue

w_q: Mean time waiting in the system.

Materials and methods

In the G/M/1/K system, the transition probability matrix is of size $(K+1)\times(K+1)$ since an arrival can find any number of customers present between zero and K inclusive. Of course, when an arrival finds K customers present, that customer is not admitted to the system, but is lost. It follows then that the matrix of transition probabilities for the embedded Markov chain is given by **William** (2009) as

$$F^{(k)} = \begin{pmatrix} 1-\beta_0 & \beta_0 & 0 & 0 & \cdots & 0 \\ 1-\sum_{i=0}^{1}\beta_i & \beta_1 & \beta_0 & 0 & \cdots & 0 \\ 1-\sum_{i=0}^{2}\beta_i & \beta_2 & \beta_1 & \beta_1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1-\sum_{i=0}^{K-1}\beta_i & \beta_{K-1} & \beta_{K-2} & \beta_{K-3} & \cdots & \beta_0 \\ 1-\sum_{i=0}^{K-1}\beta_i & \beta_{K-1} & \beta_{K-2} & \beta_{K-3} & \cdots & \beta_0 \end{pmatrix} \quad \text{(i)}$$

For example, if an arrival finds two customers already present (row 3 of the matrix), the next arrival can find three customers (with probability β_0 there were no service completions between the two arrivals), or two customers (with probability β_1 there was a single service completion between the two arrivals), or one or no customers present. The last two rows are identical because whether an arrival finds $K-1$ or K customers present, The next arrival can find any number between K and 0 according to the probabilities β_0 through $1-\sum_{i=0}^{K-1}\beta_i$.

The probabilities π_i, $i = 0, 1, \cdots, K$, which denote the probabilities of an arrival finding i customers can be obtained from the system of equations $\pi\left(F^{(K)}-1\right)=0$ by means of a reverse recurrence procedure.

The last equation is given as

$$\pi_k = \beta_0\pi_{k-1} + \beta_0\pi_k \quad \text{(ii)}$$

Which can be re-write as

$$\beta_0\pi_{k-1} + (\beta_0 - 1)\pi_k = 0 \quad \text{(iii)}$$

By assigning a value to the last component of the solution, such as $\pi_k = 1$, then we can obtain π_{k-1} as

$$\pi_{k-1} = \frac{1-\beta_0}{\beta_0} \quad \text{(iv)}$$

With the value thus computed for π_{k-1}, together with the chosen value of π_k, we can now use the second to last equation to obtain a value for π_{k-2}.

This equation namely

$$\beta_0\pi_{k-2} + (\beta_1 - 1)\pi_{k-1} + \beta_1\pi_k = 0 \quad \text{(v)}$$

Thus, π_{k-2} is given as

$$\pi_{k-2} = \frac{(1-\beta_1)\pi_{k-1}}{\beta_0} - \frac{\beta_1}{\beta_0} \quad \text{(vi)}$$

The next equation is

$$\beta_0\pi_{k-3} + (\beta_1 - 1)\pi_{k-2} + \beta_2\pi_{k-1} + \beta_2\pi_k = 0 \quad \text{(vii)}$$

Which when solved for π_{k-3}

$$\pi_{k-3} = \frac{(1-\beta_1)\pi_{k-2}}{\beta_0} - \frac{\beta_2\pi_{k-1}}{\beta_0} - \frac{\beta_2}{\beta_0} \qquad \text{(viii)}$$

In the same way, a value can be computed for all $\pi_{i\,,}\quad i = 0, 1, \cdots, K$.

At this point, a final normalization to force the sum of all $K+1$ components to be equal to 1 will yield the correct stationary probability distribution of customers at arrival epochs in a G/M/1/K queue.

Numerical example and results

Given a D/M/1/5 queue in which customers arrive at a rate of one per unit time and for which the mean of the exponential service time is equal $\frac{3}{4}$.

Solution:
We first compute the element of the transition probability matrix from the relation

$$\beta_i = \frac{\mu_i}{i\,!}e^{-\mu} = \frac{(4/3)^i}{i\,!}e^{-4/3} = \frac{(4/3)^i}{i\,!}\ 0.263597.$$

The following values of $\beta_i,\quad i = 0,\ 1,\ \cdots, 5;$ are given as

$$\beta 0 \ = \ 0.263597, \beta 1 \ = \ 0.351463, \beta 2 = 0.234309, \beta 3 \ = 0.104137, \beta 4 = \ 0.034712,$$

$$\beta 5 \ = \ 0.009257.$$

The transition probability matrix is therefore given by

$$F^{(k)} = \begin{pmatrix} 1-\beta_0 & \beta_0 & 0 & 0 & 0 & 0 \\ 1-\sum_{i=0}^{1}\beta_i & \beta_1 & \beta_0 & 0 & 0 & 0 \\ 1-\sum_{i=0}^{2}\beta_i & \beta_2 & \beta_1 & \beta_1 & 0 & 0 \\ 1-\sum_{i=0}^{3}\beta_i & \beta_3 & \beta_2 & \beta_1 & \beta_0 & 0 \\ 1-\sum_{i=0}^{K-1}\beta_i & \beta_4 & \beta_3 & \beta_2 & \beta_1 & \beta_0 \end{pmatrix}$$

Which implies that

$$F^{(k)} = \begin{pmatrix} 0.736403 & 0.263597 & 0 & 0 & 0 & 0 \\ 0.384940 & 0.351463 & 0.263597 & 0 & 0 & 0 \\ 0.150631 & 0.234309 & 0.351463 & 0.263597 & 0 & 0 \\ 0.046494 & 0.104137 & 0.234309 & 0.351463 & 0.263597 & 0 \\ 0.011782 & 0.034712 & 0.104137 & 0.234309 & 0.351463 & 0.263597 \end{pmatrix}$$

By setting $\pi_5 = 1.0$ and applying the reverse recurrence procedure to the system of equations $\pi\big(F^{(k)} - 1\big) = 0$.

This is given by

$$\pi_4 = \frac{1-\beta_0}{\beta_0} = 2.793668,$$

$$\pi_3 = \frac{(1-\beta_1)\pi_4}{\beta_0} - \frac{\beta_1}{\beta_0} = 5.540024$$

$$\pi_2 = \frac{(1-\beta_1)\pi_3}{\beta_0} - \frac{\beta_2\pi_4}{\beta_0} - \frac{\beta_2}{\beta_0} = 10.258164,$$

In the same way,

$$\pi_1 = 18.815317$$

And finally

$$\pi_0 = 34.485376,$$

The sum of all five elements is given by

$$\|\pi\|_1 = 72.892549,$$

Dividing through by this sum, we obtain the final answer as solved by **Agboola** (2010) to be

$$\pi = (0.473099,\quad 0.258124,\ 0.14073,\ 0.076003,\ 0.038326,\ 0.013719).$$

It is important to notice that the elements become excessively larger and care must be taken to preserve numerical accuracy as the size of the matrix grows. This may be accomplished by periodically renormalizing the partial solution obtained to that point.

Conclusion

In this work, transition probability matrix and steady states probabilities were obtained. Illustrative numerical examples were demonstrated on D/M/1/5 queue in which customers arrive at the rate of one per unit time and for which the mean of the exponential service time is equal to $\frac{3}{4}$, to reflects its usefulness in solving the real life problem. This gives the probability β_i, that i customers complete their service during the period k^{th} and $(k+1)^{th}$ arrivals for $\beta_i, i = 0, 1, 2, \cdots, 5$ as $\beta 0 = 0.263597, \beta 1 = 0.351463,$ $\beta 2 = 0.234309, \beta 3 = 0.104137, \beta 4 = 0.034712, \beta 5 = 0.009257$ and its corresponding stationary probability π, of an arrival finding i customers already present as $\pi = (0.473099, 0.258124, 0.14073, 0.076003, 0.038326, 0.013719)$

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