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Spectral of topological tensor product of Frechet algebras and its second dual

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Abstract

We study the spectra of tensor product of Frechet algebras A and B i.e. $A\widehat{\otimes}B$ and its bidual $(A\widehat{\otimes}B)^{//}$ .An attempt is made to show that the spectrum of the projective tensor product of Frechet Algebras $A\widehat{\otimes}B$ coincides with the spectrum of the second dual of projective tensor product of Frechet Algebras A and B i.e. $(A\widehat{\otimes}B)^{//}$

Keywords: Frechet Algebras; Second dual; Spectrum; Tensor product

1. Introduction

In this section, we give, notations, notions and definitions of concepts used in the sequel. All algebras are considered over the field of complex numbers.

1.1 Frechet algebras

1.1.1 Definition: Let X be any vector space. A real valued functional p on X is called a semi norm on X if the following conditions are satisfied:

i. $p(x) \geq 0 \quad for\ all\ x \in X.$

ii. $p(\lambda x) \leq |\lambda| p(x) \quad for\ all\ x \in X, \lambda \in \mathbb{C}.$

iii. $p(x+y) \leq p(x) + p(y) \quad for\ all\ x, y \in X$

1.1.2 **Definition**: A topological vector space X is called a locally convex space if its base of 0-neighbourhood consists of convex sets.

1.1.3 **Definition**: A topological vector space X is called a Frechet space, if it is a complete locally convex space whose topology is induced by a translation invariant metri

1.1.4 **Definition**: A topological algebra is said to be locally convex if its underlying vector space is locally convex.

1.1.5 **Definition**: A locally convex algebra A is called locally multiplicatively convex, if and only if its topology is generated by a system of continuous semi norms$(p_\propto)_{\propto \in I}$ Satisfying sub multiplicative condition.

$p_\alpha(ab) \leq p_\propto(a)p_\propto(b)$ for all $a, b \in$, A.

1.1.6 Definition: A Frechet algebra is a locally multiplicatively convex algebra which is a Frechet space. In this case the topology is generated by a countable system of semi norms $(p_j)_{j \in \mathbb{N}}$ satisfying the sub multiplications condition.

1.2 Spectrum

1.2.1. Definition: A topological dual of algebra A denoted by A^I is a continuous homomorphism from the algebra A into the complex field. The second dual of A denoted by A^{II} is the continuous homomorphism of A^I into the complex field.

1.2.2 Definition (i) A continuous character of the topological algebra A is a non-zero continuous complex morphism of A into the algebra $\mathbb{C}$ of the complexes. The set of all continuous character of A is the topological spectrum of A and is denoted by $SpecA$. Note that $SpecA$ inherits the weak* topology $\sigma(A^I, A)$ of A^I.

(ii) Let $a \in A$ and $SpecA$, the spectrum of A, the Gel'fand transform is the image of a under the Gel'fand mapping $g: A \rightarrow C(SpecA)$ which is the algebraic homomorphism of A introduced into the algebra of all complex continuous functions of spec A denoted by $\hat{a}$ and defined by $g: a \rightarrow g(a) = \hat{a} : f \rightarrow \hat{a}(f) = f(a)$, for every $a \in A$, and $f \in SpecA$. By the definition of the topology on $SpecA$, the complex function $\hat{a}$ on $SpecA$ thus defined is continuous.

1.2.3 Definition: (i) Let X and Y be locally convex spaces. A subset $H \subseteq L(X, Y)$ defined as the linear space of all functions from X to Y is called equicontinuous if for each 0-neighbourhood V in Y there-exists a 0- neighbourhood U in X such that $T(U) \subseteq Y$ for each $f \in H$..

(ii) Let A be a topological algebra, we say that A has a locally equicontinuous spectrum if for every, $f \in SpecA$ there exist a neighbourhood U of $f \in SpecA$ with respect to the relative topology of A^I such that U is an equicontinuous subset of $SpecA$.

1.3 Locally bounded approximate identity.

1.3.1. Definition: A net $(e_\propto)_{\propto \in I}$ in a topological algebra A is a right (left resp.) approximate identity if a = $\lim_{\propto} ae_\propto$ (a = $\lim_{\propto} e_\propto a\ rep.$) for all $a \in A$. A net $(e_\propto)_{\propto \in I}$ is an approximate identity if it is both a left and a right approximate identity. An approximate identity is bounded if the set $(e_\propto)_{\propto \in I}$ is bounded.

1.3.2. Definition:[4] A topological algebra A has a right (left resp.) locally bounded approximate identity if for each 0-neighbourhood $U \subset A$, there exists c > o such that for each finite subset $F \subset A$, there exists $b \in cU$ with $a - ab \in U$ ($a - ba \in U$ resp.) for all $a \in F$. A has a locally bounded approximate identity if it has both right and left locally bounded approximate identity.

1.4 Arens multiplications

1.4.1 Definition: The first and second Arens Multiplications of $\emptyset, \varphi \epsilon A^{//}$ denoted by ■ and ◊ respectively with respect to $A^/$ are defined by the formulae $(\emptyset ■ \varphi)f = \emptyset(\varphi.f)$ and$(\emptyset \ ◊\ \varphi)f = \varphi(f.\emptyset)$ where $f \in A^/$. The bilinear mapping $(\emptyset,\varphi) \longmapsto \emptyset ■ \varphi$ is separately continuous with second dual topology on $A^{//}$, therefore $(A^{//}, ■)$ with second dual topology is an associative locally convex topological algebra.

1.4.2 Definition: An element N of $A^{//}$ is called a mixed identity if N is a right identity for the first Arens Multiplication and left identity for the second Arens Multiplication. That is for each $\emptyset \in A^{//}$, $\emptyset$■N = N ◊ $\emptyset = \emptyset$.

1.5 Projective tensor product

1.5.1. Definition: The complete projective tensor product of Frechet algebras A and B is denoted by $A \widehat{\otimes} B$. The relation $r(u) = inf \sum_i p(x_i)\, q(y_i)$ where infimum is taken over the finite sum $\sum_i x_i \otimes y_i$ on $A \widehat{\otimes} B$ with $u = \sum_i x_i \otimes y_i$ defines r as a seminorm on $A \widehat{\otimes} B$.

If $U = \{x \in A: p(x) \leq 1\}$ and $V = \{y \in B: q(x) \leq 1\}$ then r is the seminorm corresponding to the convex hull $\Gamma(U \otimes V)$.

1.5.2. Proposition [5]: *The second dual of a metrisable space is a Frechet space under its strong topology*

1.5.3. Remark: (i) Let A be a Frechet algebra. By proposition 1.5.2., the second dual of the underline Frechet space is a Frechet space, hence, with Arens multiplication being associative, A^{II} is also a Frechet algebra.

1.5.4. **Lemma** [3]: *Let A and B be two given algebras and p, q sub-multiplicative semi-norms on A and B respectively. Then, "the tensor product semi-norm" $r = p \otimes q$ defined on $A \widehat{\otimes} B$ by the relation*

$r(u) = inf \sum_{i=1} p(x_i)\, q(y_i)$,

where infimum is taken over all expression of the form $u = \sum_{i=i}^{n} x_i \otimes y_i \in A\widehat{\otimes}B$, yields a sub-multiplicative semi-norm on the tensor product algebra $A\widehat{\otimes}B$.

1.5.5. Remark: Let A and B be Frechet algebras. By Lemma 1.5.4, $A\widehat{\otimes}B$ is also a Frechet algebras. Moreover, by proposition 1.5.2, the second dual of the underline Frechet space of $A\widehat{\otimes}B$, if given a strong topology is also a Frechet space. Hence, the topological algebra $(A\widehat{\otimes}B)^{//}$ with Arens multiplication is a Frechet algebra.

2 Motivation

A. Mallios [2] shown in his paper that the spectrum of tensor product of locally convex algebra ($A \otimes B$) coincides with the spectrum of the complete tensor product of locally convex algebra ($A\widehat{\otimes}B$). This motivates us to consider the relation between the spectrum of the projective tensor product of Frechet algebras A and B and the spectrum of the second dual of the projective tensor product of Frechet algebras A and B. We also consider as an assumption just as we have in the case of results of Mallios that the spectrum of the algebra considered is an equicontinuous subsets of the topological dual of the algebra.

3 Main results

The main results of this paper are presented here. The following Lemma has a direct bearing to our results.

3.1 Lemma ([3], p 49) Let A and B be two algebras and $f: A \to B$. an (algebra) morphism. Then, one has the relation $Spec_B(f(x)) \subseteq Spec_A(x)$ for every element x in A. In particular if A is a sub algebra of B then one obtains $Spec_B(x) \subseteq Spec_A(x)$ for every element x in A.

3.2. Proposition: *Let A be a Frechet algebra, $A^{//}$ its second dual and $\{e_\propto\}_\propto$ a locally bounded approximate identity in A. then, the relation spec $A^{//} = SpecA$ is valid within a continuous bijection.*

Proof: Given that $\{e_\propto\}_\propto \subset A$, let $\{x_i\}_i \subset A$ be a sequence in A. Since A has a separately continuous multiplication, we can have $l_{e_\propto}(x_i)\,) = e_\propto\, x_i$ ($l_{e_\propto}: A \longrightarrow A$). This can be extended to $A^{//}$ by using first Arens multiplication. Consider a sub-sequence $((x_i)_\beta)_{\beta \in I}$ of $\{x_i\}$ in A, we have by definition $\lim_{\beta} \lim_{\propto} (x_i)_\beta\ e_\propto = x_i \blacksquare \mathrm{N} = x_i$,

where N is an identity element in $A^{//}$. Therefore $x_i \in A^{//}$. Define non zero continuous complex morphisms $j: A^{II} \longrightarrow \mathbb{C}$ and $k: A \longrightarrow \mathbb{C}$ respectively where their sets of continuous characters $SpecA^{II}$ and $SpecA$ are induced by the weak topologies $\sigma\,(A^{II}, A^{I})$ and $\sigma\,(A^{I}, A)$ respectively. Since A is a Frechet algebra, there exists a continuous Gel'fand map $\widehat{x_\iota}\,(k) = k\,(x_i)$ where $x_i \in A$ such that $k \in SpecA$ and this $\Longrightarrow \widehat{x_\iota} \in C\,(SpecA)$.

So also, since A^{II} is also a Frechet algebra there exist a continuous Gel'fand map $\widehat{x_\iota}\,(j) = j\,(x_i)$ where $x_i \in A^{II}$ such that $j \in SpecA^{II} \implies \widehat{x_\iota} \in C\,(SpecA^{II})$. Hence, $\widehat{x_\iota} \in C\,(SpecA)$ implies that $\widehat{x_\iota} \in C\,(SpecA^{II})$, therefore $C\,(SpecA) \subset C\,(SpecA^{II})$. Since the continuities of $\widehat{x_\iota}$on $SpecA$ and $\widehat{x_\iota}$ on $SpecA^{II}$ depend on the topologies on $SpecA$ and on $SpecA^{II}$, hence, $SpecA \subset SpecA^{II}$ ……………….(*)

Conversely, let the sequence x_i in A^{II} converge to $z \in A^{II}$. Since $\widehat{x_\iota}$ is continuous on $SpecA^{II}$, there exists $\widehat{x_\iota} \longrightarrow \hat{z}$ uniformly continuous on $SpecA^{II}$. By local theory of spectrum, we can have $Spec_{A^{II}}(z) \in (A^{II})^I$ .Considering an inclusion mapping $i^I: (A^{II})^I \longrightarrow A^I$ which induces a map $i^I_*: Spec_{A^{II}}(z) \longmapsto Spec_A(z)$.. This then implies that $Spec_{A^{II}}(z) \subset Spec_A(z)$ by Lemma 3.1. This subsequently implies that $SpecA^{II} \subset SpecA$……….(**).

Combining (*) and (**), we have $SpecA = SpecA^{II}$ valid within a continuous bijection $i^I_*: SpecA^{II} \longrightarrow SpecA$. ∎

3.3 Theorem: *Let A and B be Frechet algebras with locally bounded approximate identities and A^{II} and B^{II} their second duals respectively. Let $A\hat{\otimes}B$ and $(A\hat{\otimes}B)^{II}$ be the projective tensor product of Frechet algebras A, B and second dual of projective tensor product of Frechet algebras A and B respectively. Then Spec($A\hat{\otimes}B$) = $Spec(A\hat{\otimes}B)^{II}$ up to a homeomorphism*.

Proof:

We first show that the mapping of $Spec(A\hat{\otimes}B)$ onto $SpecA^{II}\hat{\otimes}SpecB^{II}$ is continuous. By the universal property of tensor product, the map

$f: SpecA^{II} \times SpecB^{II} \longrightarrow SpecA^{II}\hat{\otimes}SpecB^{II}$ is continuous. By proposition 3.2, $SpecA\hat{\otimes}SpecB = SpecA^{II}\hat{\otimes}SpecB^{II}$ and by ([2], Theorem 4:2),

$Spec(A\hat{\otimes}B)$= $SpecA\hat{\otimes}SpecB$. Hence $Spec(A\hat{\otimes}B)$ = $SpecA^{II}\hat{\otimes}SpecB^{II}$ as a subset of $(A^{II})^I\hat{\otimes}\ (B^{II})^I$. Let $\emptyset_i \otimes \varphi_i$ be a net in $(A\hat{\otimes}B)^{II}$ .Then there exist a net

$\widehat{\phi_\iota\otimes\varphi_\iota} \in C(Spec(A\hat{\otimes}B)^{II})$ that converges to $\widehat{\phi\otimes\varphi} \in C(Spec(A\hat{\otimes}B)^{II})$ as a Gel'fand map and by its continuity on $Spec(A\hat{\otimes}B)^{II}$. Also by separately continuous multiplication in the Frechet algebra $(A\hat{\otimes}B)^{II}$, we have $l_{a\otimes b}\ (A\hat{\otimes}B)^{II} \to A^{II}\hat{\otimes}B^{II}$ defined by

(a⊗b) ∎ $(\emptyset_i \otimes \varphi_i) \longmapsto$ (a∎$\emptyset_i$) ⊗ (b∎φ_i) and converges in the weak * topology to

(a∎$\emptyset$) ⊗ (b∎φ). Hence, there exist by Gel'fand transformation on $Spec(A\hat{\otimes}B)^{II}$ the continuous map $\widehat{l^{II}_{a\otimes b}}: C(Spec(A\hat{\otimes}B)^{II}) \to C\left(Spec\ A^{II}\hat{\otimes}SpecB^{II}\right)$ induced by $l_{a\otimes b}$ and defined by

$\left(\widehat{a\otimes b}\right)\cdot\left(\widehat{\phi_\iota\otimes\varphi_\iota}\ \right) \mapsto (\widehat{a\cdot\phi_\iota})\otimes\left(\widehat{b\cdot\varphi_\iota}\right)$ and converges in the weak * topology induced on $SpecA^{II}\hat{\otimes}SpecB^{II}$ by $(A^{II})^I\hat{\otimes}\ (B^{II})^I$ $to\ \left(\widehat{a\cdot\phi}\right)\otimes\left(\widehat{b\cdot\varphi}\right)$ as a result of $\widehat{\phi_\iota} \longrightarrow \hat{\phi}$ on $SpecA^{II}$ and $\widehat{\psi_\iota} \longrightarrow \hat{\psi}$ on $SpecB^{II}$. Hence, this implies that

$Spec(A\hat{\otimes}B)^{II} \subset SpecA^{II}\hat{\otimes}SpecB^{II}$ ………………………(i)

Conversely, by ([1], Remark 1.5, p7) $(A^{II})^I\hat{\otimes}$ $(B^{II})^I$is considered as a linear subspace of $((A\hat{\otimes}B)^{II})^I$. Hence, $SpecA^{II}\hat{\otimes}SpecB^{II} \subset Spec(A\hat{\otimes}B)^{II}$ ………………..(ii).

Combining (i) and (ii) gives $Spec(A\hat{\otimes}B)^{II} = SpecA^{II}\hat{\otimes}SpecB^{II}$.

So also since $Spec(A\hat{\otimes}B)$ = $SpecA^{II}\hat{\otimes}SpecB^{II}$.There fore, $Spec\left(A\hat{\otimes}B\right) = Spec(A\hat{\otimes}B)^{II}$. Hence, $Spec(A\hat{\otimes}B)^{II}$ and $Spec(A\hat{\otimes}B)^{II}$ coincide as topological spaces. ∎

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