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Perturbation iteration transform method for solving Benjamin-Bona-Mahony equation.

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Abstract

The Perturbation Iteration Transform Method (PITM), a combination of Laplace Transform method and Perturbation Iteration Algorithm is applied to solve Benjamin-Bona-Mahony (BBM) equation. The numerical solution of the equation, obtained using the initial condition, when compared with solutions obtained through Variational Homotopy Perturbation Method (VHPM) and with the exact solution is more accurate. This shows that the method is efficient and more reliable.

Keywords: Perturbation Iteration Technique, Laplace Transform, Perturbation Iteration Transform Method, Benjamin-Bona-Mahony Equation.

1. Introduction

The Benjamin-Bona-Mahony (BBM) equation, also known as the regularized long-wave equation is the partial differential equation

$$u_t - u_{xxt} + u_x + uu_x = 0 \qquad (1.1)$$

The equation was introduced in (Benjamin T.B.,*et al*, 1972), as an improvement of the Korteweg-de Vries (Kdv) equation for modeling long waves of small amplitude in 1+1 dimensions.

Equation (1.1) is related to generalized Benjamin-Bona-Mahony Equation given by

$$u_x - u_{xxt} + u_t - \alpha u_{xx} + u_x = 0; \quad 0 \le \alpha \le 1\, ; n \in N \qquad (1.2)$$

with $\alpha = 0$ *and* $n = 1$. $u(x,t)$ is the solution of equation (1.1).

The Numerical and Analytical approximations of non-linear partial differential equations (NLPDE), usually encountered by Mathematicians, Physicists and Engineers have been of considerable interest in recent times. (Easif *et al*, 2015) applied Variational Homotopy Perturbation Method (VHPM) to solve BBM equation (Easif *et al*, 2015) and (Hasan *et al*, 2011) compared Homotopy Perturbation Method (HPM) and Adomian Decomposition Method (ADM) for the solution of non-linear BBM equation (Hasan *et al*, 2011).

In this article, Perturbation Iteration Transform Method (PITM) is used to solve the nonlinear equation (1.1). The main goal of the paper is to find the approximate solution of the BBM by the PITM which has already been successfully applied to several non-linear problems like Linear and Non-linear Klein-Gordon Equations (Khalid *et al*, 2016), Newell-Whitehead-Segel model Equations (Akinlabi and Edeki, 2017), Bratu-type Equations Aksoyand Pakdemirli (2010), and so on.

2. Perturbation iteration algorithm (Aksoy and Pakdemirli, 2010)

Consider the development of a perturbation algorithm, named PIA (1,1), by using a correction term in the perturbed expansion and correction terms of only the first derivatives in the Taylor's series expansion $n = 1, m = 1$ (say).

Now, consider a second-order differential equation:

$$F(\dot{u}, u'', u, \varepsilon) = 0 \qquad (2.1)$$

where,

$$u = u(x,t), \dot{u} = \frac{\partial u}{\partial t}, u'' = \frac{\partial^2 u}{\partial x^2}$$

and ε is the perturbation parameter.

Considering only one correction term in the perturbation expansion

$$u_{n+1} = u_n + \varepsilon (u_c)_n \qquad (2.2)$$

Substituting equation (2.2) in equation (2.1) and expanding in a Taylor series with first derivatives only gives:

$$\mathrm{F}(\dot{u}, u'', \mathrm{u}, 0) + F_{\dot{u}}(\dot{u}, u'', u, 0)\varepsilon(\dot{u}_c)_n + F_{u''}(\dot{u}, u'', u.0)\varepsilon(u_c'')_n + F_u(\dot{u}, u'', u, 0)\varepsilon(u_c)_n + F_\varepsilon(\dot{u}, u'', u, 0)\varepsilon = 0 \qquad (2.3)$$

where, $u = u(x,t), F_{\dot{u}} = \frac{\partial F}{\partial \dot{u}}, F_{u''} = \frac{\partial F}{\partial u''}, F_u = \frac{\partial F}{\partial u}, F_\varepsilon = \frac{\partial F}{\partial \varepsilon}$ and ε the perturbation parameter to be evaluated at $\varepsilon = 0$.

If we reorganize equation (2.3), we obtain:

$$(\dot{u}_c)_n + \frac{F_{u''}}{F_{\dot{u}}}(u_c'')_n = -\frac{F_\varepsilon + \frac{F}{\varepsilon}}{F_{\dot{u}}} - \frac{F_u}{F_{\dot{u}}}(u_c)_n \qquad (2.4)$$

With initial guess u_0, the term $(u_c)_0$ is obtained from equation (2.4) and then inserted in equation (2.2) to evaluate u_1. We continue the iterative process using equation (2.4) and equation (2.2) until satisfactory result is obtained.

3. Perturbation Iteration transform method (Khalid et al., 2016)

To illustrate the basic idea of this method, we consider a general nonlinear PDE with boundary conditions of the form:

$$Eu(x,t) + Fu(x,t) + Gu(x,t) + Hu(x,t) = k(x,t) \qquad (3.1)$$

with initial condition

$$u(x,0) = r(x) \qquad (3.2)$$

where, $E = \frac{\partial}{\partial t}$ is the first order linear differential operator, $F = \frac{\partial^2}{\partial x^2}$ is the second order linear differential operator, G, H are linear and non-linear terms and $k(x,t)$ is the source term.

Taking the Laplace Transform of both sides of equation (3.1), we obtain

$$L[Eu(x,t)] + L[Fu(x,t)] + L[Gu(x,t)] + L[Hu(x,t)] = L[k(x,t)] \qquad (3.3)$$

Using the differential property of the Laplace Transform we obtain

$$L[Eu(x,t)] = \frac{r(x)}{s} + \frac{1}{s}L[k(x,t)] - \frac{1}{s}L[Fu(x,t) + Gu(x,t) + Hu(x,t)] \qquad (3.4)$$

Operating with the inverse Laplace Transform on both sides of equation (3.4) leads to

$$u(x,t) = P(x,t) - L^{-1}\left\{\frac{1}{s}L[Fu(x,t) + Gu(x,t) + Hu(x,t)]\right\} \qquad (3.5)$$

where, $P(x,t)$ is the term arising from the source term and the given initial condition.

On using the Perturbation Iteration Transform Method, equation (3.5) results to:

$$u(x,t) - P(x,t) + u_c(x,t)\varepsilon + L^{-1}\left\{\frac{1}{s}L[Fu(x,t) + Gu(x,t) + Hu(x,t)]\right\}\varepsilon = 0 \quad (3.6)$$

Thus:

$$u_c(x,t) = \frac{P(x,t) - u(x,t)}{\varepsilon} - L^{-1}\left\{\frac{1}{s}L[Fu(x,t) + Gu(x,t) + Hu(x,t)]\right\} \quad (3.7)$$

This is a coupling of the Laplace Transform Method and the Perturbation Iteration Algorithm. The initial point $(u_c)_0$ is calculated from equation (3.7) and then substituted in equation (2.2) to obtain u_1. The iterative procedure is repeated until a satisfactory result is obtained.

The approximate solution is thus obtained as:

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + u_4(x,t) + \ldots \quad (3.8)$$

4. Numerical example

Consider the non-linear Benjamin-Bona-Mahony equation (Easif *et al*, 2015) :

$$u_t - u_{xxt} + u_x + uu_x \quad (4.1)$$

with initial condition

$$u(x,0) = sech^2\left(\frac{x}{4}\right) \quad (4.2)$$

and exact solution

$$u(x,t) = sech^2\left(\frac{x}{4} - \frac{t}{3}\right) \quad (4.3)$$

Applying Laplace Transform on both sides of equation (4.1) subject to the initial condition of equation (4.2), we obtain

$$L[u(x,t)] = \frac{1}{s}sech^2\left(\frac{x}{4}\right) + \frac{1}{s}L[u_{xxt} - u_x - uu_x] \quad (4.4)$$

The Inverse of Laplace Transform implies that

$$u(x,t) = sech^2\left(\frac{x}{4}\right) + L^{-1}\left\{\frac{1}{s}L[u_{xxt} - u_x - uu_x]\right\} \quad (4.5)$$

Now, applying the Perturbation Iteration Method to equation (4.5), we obtain

$$u(x,t) - sech^2\left(\frac{x}{4}\right) + u_c(x,t)\varepsilon - L^{-1}\left\{\frac{1}{s}L[u_{xxt} - u_x - uu_x]\right\}\varepsilon = 0 \quad (4.6)$$

Thus

$$u_c(x,t) = \frac{-u(x,t) + sech^2(\frac{x}{4})}{\varepsilon} + L^{-1}\left\{\frac{1}{s}L[u_{xxt} - u_x - uu_x]\right\} \quad (4.7)$$

We have

$$u_0(x,t) = sech^2\left(\frac{x}{4}\right)$$

$$u_1(x,t) = L^{-1}\left\{\frac{1}{s}L[u_{0xxt} - u_{0x} - u_0u_{0x}]\right\} \quad (4.8)$$

This gives

$$u_1(x,t) = t\left(\frac{1}{2}sech^2\left(\frac{x}{4}\right)\tanh\left(\frac{x}{4}\right) + \frac{1}{2}sech^4\left(\frac{x}{4}\right)\tanh\left(\frac{x}{4}\right)\right) \quad (4.9)$$

Also

$$u_2(x,t) = L^{-1}\left\{\frac{1}{s}L[u_{1xxt} - u_{1x} - u_1u_{1x}]\right\} \quad (4.10)$$

And

$$\begin{aligned} u_2(x,t) = {} & t\left(\frac{1}{8}sech^2\frac{x}{4}tanh^3\left(\frac{x}{4}\right) - \frac{1}{4}sech^4\left(\frac{x}{4}\right)\tanh\left(\frac{x}{4}\right) + \frac{1}{2}sech^4\left(\frac{x}{4}\right)tanh^3\left(\frac{x}{4}\right) - \frac{7}{16}sech^6\left(\frac{x}{4}\right)\tanh\left(\frac{x}{4}\right)\right) \\ & + t^2\left(\frac{1}{4}sech^2\left(\frac{x}{4}\right)tanh^2\left(\frac{x}{4}\right) - \frac{1}{8}sech^4\left(\frac{x}{4}\right) + \frac{1}{2}sech^4\left(\frac{x}{4}\right)tanh^2\left(\frac{x}{4}\right) - \frac{1}{8}sech^6\left(\frac{x}{4}\right)\right) \\ & + t^3\left(\frac{1}{8}sech^4\left(\frac{x}{4}\right)tanh^3\left(\frac{x}{4}\right) - \frac{1}{16}sech^6\left(\frac{x}{4}\right)\tanh\left(\frac{x}{4}\right) + \frac{3}{8}sech^6\left(\frac{x}{4}\right)tanh^3\left(\frac{x}{4}\right) - \frac{1}{8}sech^8\left(\frac{x}{4}\right)\tanh\left(\frac{x}{4}\right) + \frac{1}{4}sech^8\left(\frac{x}{4}\right)tanh^3\left(\frac{x}{4}\right) - \frac{1}{16}sech^{10}\left(\frac{x}{4}\right)\tanh\left(\frac{x}{4}\right)\right) \end{aligned} \quad (4.11)$$

Other components of the PITM can be determined in a similar way.
Therefore the approximate solution of the BBM equation (4.1) is given by

$$u(x,t) \cong u_0(x,t) + u_1(x,t) + u_2(x,t) + \dots \quad (4.12)$$

5. Tables

Table 1. Absolute error between PITM and exact solution

	t = 0	t = 0.5	t = 1
x = -5	0	5.59868e-004	7.4985968e-003
x = 0	0	3.52179339e-002	1.485445147e-001
x = 5	0	1.57263145e-002	5.63910317e-002

Table 2. Absolute error between PITM and exact solution

	t = 0	t = 2.5	t = 5
x = 0	0	1.097043162e+000	6.954523403e-001
x = 25	0	1.73780592e-005	2.633947552e-004
x = 50	0	0.647712351e-010	9.818953446e-010

Table 3. Comparing absolute error of VHPM (Easif *et al*, 2015) and present method with exact solution

	x =-5	x = -5	x = 0	x = 0
	VHPM	PITM	VHPM	PITM
t = 0	5.551115123125783e-017	0	7.4985968e-003	0
t = 0.5	7.314427490900022e-002	5.59868e-004	1.485445147e-001	3.52179339e-002
t = 1	0	7.4985968e-003	5.63910317e-002	1.485445147e-001

Table 4. Comparing absolute error of VHPM (Easif *et al*, 2015) and present method with exact solution

	x =5	x = 5
	VHPM	PITM
t = 0	5.551115123125783e-017	0
t = 0.5	6.534308332294098e-002	1.57263145e-002
t = 1	1.230071227985541e-001	5.63910317e-002

Table 5. Comparing absolute error of VHPM (Easif *et al*, 2015) and present method with exact solution

	x =0	x = 0	x = 25	x = 25
	VHPM	PITM	VHPM	PITM
t = 0	0	0	0	0
t = 2.5	1.097018849581008e+000	1.097043162e+000	4.770884185166814e-005	1.73780592e-005
t = 5	5.383034896646911e+000	6.954523403e-001	3.469615990472011e-004	2.633947552e-004

Table 6. Comparing absolute error of VHPM (Easif *et al*, 2015) and present method with exact solution

	x =50	x = 50
	VHPM	PITM
t = 0	0	0
t = 2.5	1.778067925425482e-010	0.647712351e-010
t = 5	1.293335597251119e-009	9.818953446e-010

6. Conclusions

In this study, we successfully demonstrated the use of perturbation iteration transform method to solve the Benjamin-Bona-Mahony equation. The results of the method is better when compared with the VHPM solutions. The method works well with strongly nonlinear systems and small parameter restriction associated with standard perturbation methods are removable with PITM. Furthermore,

(i) As shown in Table 1, the error decreases when the time is small in a small space.

(ii) As shown in Table 2, the error increases when time increases in a wide space.

(iii) As shown in Table 3-6, PITM is more accurate than VHPM.

(iv) In general, whenever a space gets extended the error decreases and gets closer to zero.

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