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Comparative methods of asymptotic sequence of multiple-scale analysis of partial differential equation

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Abstract

Multiple–scale analysis comprises of techniques used to construct uniformly valid approximations to the solutions of perturbation problem in which the solution depends simultaneously on widely different scales. We considered the derivative expansion method and two-variable methods. Examples were used in performing Numerical experiment to ascertain the effectiveness, accuracy, reliability and stability of the methods

Keywords: Asymptotic Sequence, Multiple Scale, Partial Differential Equations(PDE)

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Introduction

The differential equations encountered in applied mathematics, science, and engineering research is only rarely soluble in terms of familiar mathematical functions. When an exact solution is lacking, it is often desirable to use local analysis to determine the approximate behavior of a solution near a point of interest (which could even be ∞). Asymptotic series provide a powerful technique for constructing such approximations.

The method of multiple scales (also called the multiple-scale analysis) comprises of techniques used to construct uniformly valid approximations to the solutions of perturbation problems in which the solutions depend simultaneously on widely different scales. This is done by introducing fast-scale and slow-scale variables for an independent variable, and subsequently treating these variables, fast and slow, as if they are independent[10].

Scaling and self-similarity are very useful mathematical concepts, having many applications in science and technology. Scaling laws provide evidence of the existence of hidden structures for dynamical systems; self-similarity manifests itself in the study of fractals, chaotic systems, incompressible fluids, etc. Apart from being a way to get exact solutions, scaling symmetries are essential for defining the correct variables necessary to describe the universal features of many models, for instance their asymptotic behaviour, as happens in the renormalization of quantum theories or in the multiple-scale expansions [4].

In the last years, new physical and mathematical motivations have emerged for the study of dynamical models on a lattice. Many systems of interest in nonlinear optics or in relation with DNA are indeed discrete. Also, awareness is growing that space-time may possess an intrinsic discreteness[24].

For continuous systems, Lie group theory is the most efficient approach to get self-similar solutions [26]. A direct extension of Lie theory to lattice equations has not been yet as fruitful, as its continuous counterpart, in the study of scaling symmetries[21]. For lattice models, we can only define very few group transformations, that is, we can add a constant (finite translation) or multiply or divide a discrete variable by an integer multiple (finite dilation). These are fundamentally discrete symmetries. In principle, the infinitesimal counterparts of these transformations do not exist, although they can be formally introduced.

Among the perturbation methods used to study asymptotic of problems involving a small parameter, very important ones are the reductive multiple-scale expansions [5]. They are relevant when ordinary perturbation methods fail to give accurate uniform approximations of solutions. One introduces multiple scales to avoid secularity terms and obtain uniform asymptotic. In the case of continuous systems, this procedure has been intensively investigated for the wideness of physical and engineering applications [5].

Theoretically, the multiple-scale approach has given important results, especially in the study of the integrability of nonlinear partial differential equations. Indeed, while solving an integrable system, when performing multiple-scale expansion solution to one leads to other integrable systems [30, 34]. In [7, 8], Calogero and Eckhaus have proven that a necessary condition for an equation to be integrable is that its multi-scale reduction be integrable as well. In [11] higher order corrections have been also investigated: it has been shown that, under suitable hypotheses, the leading amplitude modulations of one-dimensional strongly dispersive waves satisfy the nonlinear Schrodinger hierarchy.

The general problem of the multiple-scale analysis for discrete systems seems to be technically much more tricking [22], [18], [15], [1]. In [22], a discretization of the standard reductive method has been proposed, relying on the definition of a large grid of points, indexed by slow variables, and on the comparison of the different physical observable between the original and the large scale grid. This procedure has been further studied in [18] and in [15], where an integrability test based on discrete multiple-scale analysis has been proposed for a Z^2-lattice. We observe that in general the process of rescaling is not univocally defined, since the discrete derivative can be defined in infinitely many different ways. More important, a multiple-scale expansion usually leads to discrete equations of infinite order.

The paper is subdivided into five sections; section one is an introductory part, section two is the preliminaries/definition of Asymptotic expansion, while section three is the analysis of the two methods of the Multiple-scale; section four is the Numerical Examples/results; and section five is the concluding part of the paper.

1. Asymptotic expansion definition

Definition 1:[16] An *asymptotic sequence* at *a* is a sequence of functions $(\phi_k)_{k=1}^{\infty}$ such that $\phi_{k+1}(x) = o(\phi_k(x))$ as $x \to a$, for every *k*. If *a* is finite, an obvious example is the powers of (*x* - *a*), since

$(x-a)^{k+1} = o((x-a)^k)$ as x → a

We can start this sequence at *k* <0 if we so desire. Other possibilities include a sequence of fractional powers, $x^{\alpha+k\beta}$ or log /x/; log/j log /x//; : : :, or

$$\left(\frac{x^k}{1+x}\right)_{k=K}^{\infty}$$

and so on.

If $a = \infty$, then

$$x^n, x^{n+1}, \ldots, x, 1, \frac{1}{x}, \frac{1}{x^2}, \ldots$$

as

$$\frac{1}{x}, \frac{1}{x(x+1)}, \frac{1}{x(x+1)(x+2)}, \ldots,$$

and

$$e^{\alpha_1 x}, e^{\alpha_2 x}, e^{\alpha_3 x}, \ldots$$

for any strictly decreasing sequence $(\phi_k)_{k=1}^{\infty}$.

Also valid is something as irregular as

$$2x, \frac{1}{x}, \frac{1}{(1+x)^2}, \frac{1}{x^{5/2}}, \ldots,$$

provided subsequent terms also follow the definition.

Definition 2:[16] Let *f* be a function on a domain with *a* as a limit point (that is, either *a* is inside the domain, or is a boundary point). *f* is said to have an *N-term asymptotic expansion at a* in terms of (ϕ_k) if there are constants $(\phi_k)_{k=1}^{N}$ such that

$$f(z) = \sum_{k=1}^{N} a_k\phi_k(z) + o(\phi_N(z)) \qquad as \quad x \to a$$

If this relationship holds for any positive integer *N*, the series $\sum_{k=1}^{N} a_k\phi_k(z)$ is called an *asymptotic expansion of* ***f*** *at* ***a***, and we write

$$f(z) \approx \sum_{k=1}^{\infty} a_k\phi_k(z) \qquad as \quad x \to a$$

This is one of those definitions that look completely useless which gives us no way to find an asymptotic expansion.

Remark:

Here are some properties of this definition:

i. An asymptotic expansion may not converge, for any value of the variable *z*

ii. Obviously this definition is linear, so if *f*; *g* have asymptotic expansions

$$f(z) \approx \sum_{k=1}^{\infty} a_k\phi_k(z) \qquad as \quad x \to a$$

$$g(z) \approx \sum_{k=1}^{\infty} b_k\phi_k(z) \qquad as \quad x \to a$$

Then $\alpha f(z) + \beta g(z)$ has asymptotic expansion

$$\alpha f(z) + \beta g(z) \approx \sum_{k=1}^{\infty} (\alpha a_k + \beta b_k)\phi_k(z) \qquad as \quad x \to a$$

for any constants *α, β.*

iii. A function has different asymptotic expansions with respect to different asymptotic sequences: for example, let $f(z) = 1 + z^{-1}$. Then with respect to the sequence $(z^{-k})_{k=0}^{\infty}$ $at \propto$

$$f(z) \approx 1.1 + 1.\frac{1}{x} + 0.\frac{1}{x^2} + \ldots,$$

Whereas with respect to the sequence $(1 + z)^{-k}$, we have

$$1 + \frac{1}{z} = \frac{1+z}{(1+z)-1} = \left(1 - \frac{1}{1+z}\right)^{-1}$$

and so,

$$1 + \frac{1}{z} \approx \sum_{k=0}^{\infty} \frac{1}{(1+z)^k}$$

iv. An asymptotic expansion does not uniquely determine a function: for example,

$$x \uparrow \infty, (1+x)^{-1} \quad and \quad (1+e^{-x})(1+x)^{-1}$$

both have asymptotic expansion

$$\sum_{k=0}^{\infty} (-1)^k z^{-(k+1)}$$

Proposition:[10] *Given f and an asymptotic sequence* (ϕ_k) *at a, if f has an asymptotic expansion*

$$f(z) \approx \sum_{k=1}^{\infty} a_k \phi_k(z) \qquad as \quad x \to a,$$

this expansion is unique.

Proof. We may proceed by an inductive argument. We are looking for an expansion of the form

$$f(z) = \sum_{k=1}^{N-1} a_k \phi_k(z) + a_N \phi_N(z) + R_N(z)$$

where $R_N(z) = o(\phi_N(z)) \quad as \quad x \to a.$ Then

$$\frac{f(z) - \sum_{k=1}^{N-1} a_k \phi_k(z)}{\phi_N(z)} = a_N + \frac{R_N(z)}{\phi_N(z)}$$

and since the latter term on the right tends to 0 as $z \to \infty$, we have a unique determination for a_N:

$$a_N = \lim_{z \to a} \frac{f(z) - \sum_{k=1}^{N-1} a_k \phi_k(z)}{\phi_N(z)}$$

(The proof shows that limit must exist, from the definition of an asymptotic series, and hence the definition of R_N.) This works for any N, including the case $N = 1$, where the limit is just $f(z)/\phi_1(z)$.

2. Development of Methods of Asymptotic Sequence of Multiple-scale Analysis

2.1 Derivative Expansion Method

The derivative expansion method is probably the most common of the various multiple scale methods. We introduces several time (length) scales and treats them as independent variables of the form:

If **t** is the variable and **ε** is the small parameter, introduce the auxiliary time scales

$\tau_1 = \varepsilon t,\ \tau_2 = \varepsilon^2 t, \ldots, \tau_N = \epsilon^N t$ and express $u(t,\varepsilon) = u(t, \tau_1, \ldots, \tau_N, \varepsilon)$. Then

$$u' = \frac{\partial u}{\partial t} = \left(\frac{\partial}{\partial t} + \varepsilon \frac{\partial}{\partial \tau_1} + \varepsilon^2 \frac{\partial}{\partial \tau_2} + \ldots + \varepsilon^N \frac{\partial}{\partial \tau_N} \right) u$$

Now, consider the derivative expansion method to the problem

$$\begin{cases} u'' + 2\varepsilon u' + u = 0 & t \in [0, \infty), \\ u(0,\varepsilon) = 1, & u'(0,\varepsilon) = 0, \end{cases}$$

On solving, introduce new time scales $\tau_1 = \varepsilon t,\ \tau_2 = \varepsilon^2 t$ and let $u(t,\varepsilon) = u(t,\ \tau_1, \tau_2, \varepsilon)$,

Then

$$u' = \frac{\partial u}{\partial t} + \varepsilon \frac{\partial u}{\partial \tau_1} + \varepsilon^2 \frac{\partial u}{\partial \tau_2}$$

$$u'' = \left(\frac{\partial}{\partial t} + \varepsilon \frac{\partial}{\partial \tau_1} + \varepsilon^2 \frac{\partial}{\partial \tau_2} \right)^2 u$$
$$= \frac{\partial^2 u}{\partial t^2} + 2\varepsilon \frac{\partial^2 u}{\partial t \partial \tau_1} +$$
$$\varepsilon^2 \left(2 \frac{\partial^2 u}{\partial t \partial \tau_2} + \frac{\partial^2 u}{\partial \tau_1^2} \right) + \sigma(\varepsilon^2)$$

The problem then become

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} + 2\varepsilon \dfrac{\partial^2 u}{\partial t \partial \tau_1} + \varepsilon^2 \left(2 \dfrac{\partial^2 u}{\partial t \partial \tau_2} + \dfrac{\partial^2 u}{\partial \tau_1^2} \right) + u + \sigma(\varepsilon^2) = 0. \\ u(0,\ 0,\ 0,\ \varepsilon) = 1. \\ \dfrac{\partial u}{\partial t}(0,\ 0,\ 0,\ \varepsilon) + \varepsilon \dfrac{\partial u}{\partial \tau_1}(0,\ 0,\ 0,\ \varepsilon) + \varepsilon^2 \dfrac{\partial u}{\partial \tau_2}(0,\ 0,\ 0,\ \varepsilon) = 0 \end{cases}$$

we look for a solution to this partial differential equation of the form

$$u(t\ \tau_1,\ \tau_2,\ \varepsilon) = u_0(t, \tau_1, \tau_2) + \varepsilon u_1(t, \tau_1) + \varepsilon^2 u_2(t) + \sigma(\varepsilon^2)$$

we find

$$\varepsilon^0 : \begin{cases} \dfrac{\partial^2 u_0}{\partial t^2} + u_0 = 0 \\ u_0(0,\ 0,\ 0) = 1 \\ \dfrac{\partial u_0}{\partial t}(0,\ 0,\ 0) = 0 \end{cases}$$

$$\varepsilon^1 : \begin{cases} \dfrac{\partial^2 u_1}{\partial t^2} + u_1 = -2\dfrac{\partial^2 u_0}{\partial t \partial \tau_1} - 2\dfrac{\partial u_0}{\partial t} \\ u_1(0,\ 0) = 0 \\ \dfrac{\partial u_1}{\partial t}(0,\ 0) = -\dfrac{\partial u_0}{\partial \tau_1}(0,\ 0,\ 0) \end{cases}$$

$$\varepsilon^2 : \begin{cases} \dfrac{\partial^2 u_2}{\partial t^2} + u_2 = -2\dfrac{\partial^2 u_0}{\partial t \partial \tau_2} - \dfrac{\partial^2 u_0}{\partial \tau_1^2} - 2\dfrac{\partial u_0}{\partial \tau_1} - 2\dfrac{\partial^2 u_1}{\partial t \partial \tau_1} - 2\dfrac{\partial u_0}{\partial t} \\ u_2(0) = 0 \\ \dfrac{\partial u_2}{\partial t}(0) = -\dfrac{\partial u_0}{\partial \tau_2}(0,0,0) - \dfrac{\partial u_1}{\partial \tau_1}(0,0) \end{cases}$$

The solution to the ε^0 problem is

$$u_0(t,\ \tau_1,\ \tau_2) = A(\tau_1,\ \tau_2) Cos\ t + B(\tau_1,\ \tau_2) Sin\ t.$$

When the initial conditions on u_0 implies that A(0, 0) = 1 and B(0, 0) = 0, and ε^1 problem becomes,

$$\frac{\partial^2 u_1}{\partial t^2} + u_1 = 2\left(\frac{\partial A}{\partial \tau_1} + A\right) Sint - 2\left(\frac{\partial B}{\partial \tau_1} + B\right) Cost$$

The only way to avoid secularities in solutions to this sinusoidal differential equation, which is being driven at its natural frequency, is to use our freedom in choosing the functions A and B to insist that

$$\frac{\partial A}{\partial \tau_1} + A = 0$$

$$\frac{\partial B}{\partial \tau_1} + B = 0$$

We thus find

$$A(\tau_1, \tau_2) = \alpha(\tau_2) e^{-\tau_1}$$

$$B(\tau_1, \tau_2) = \beta(\tau_2) e^{-\tau_1}$$

where the initial conditions A(0, 0) = 1 and B(0, 0) = 0 imply that $\alpha(0) = 1$ and $\beta(0) = 0$. With this choice of A and B, we can now solve for both u_0 and u_1:

$$u_0(t,\ \tau_1, \tau_2) = e^{-\tau_1}[\alpha(\tau_2) Cost + \beta(\tau_2) Sint],$$

$$u_1(t,\ \tau_1) = C(\tau_1) Cost + D(\tau_1) Sint]$$

The initial conditions on u_1 which implies that C(0) = 0 and D(0) = $\alpha(0)$ = 1, and ε^2 equation becomes

$$\frac{\partial^2 u}{\partial t^2} + u_2 = [(2\alpha' + \beta)e^{-\tau_1} + 2(C' + C)Sint] + [(-2\beta' + \alpha)e^{-\tau_1} - 2(D' + D)Cost]$$

Again, we used to remove the secular terms

$$(2\alpha' + \beta) + 2e^{\tau_1}(C' + C) = 0$$

$$(-2\beta' + \alpha) - 2e^{-\tau_1}(D' + D) = 0$$

We note that the terms involving α and β are functions of τ_2 only, while the terms involving C and D are functions of τ_1 only. Hence $2\alpha' + \beta = -2e^{\tau_1}(C' + C)$ and $(-2\beta' + \alpha) = 2e^{\tau_1}(D' + D)$ must be constants. For simplicity, we choose these constants to be zero. The system

$$2\alpha' + \beta = 0$$

$$-2\beta' + \alpha = 0$$

Also, initial conditions C(0) = 0 and D(0) = 1 which implies that $C(\tau_1) = 0$ and $D(\tau_1) = e^{-\tau_1}$

Thus

$$u_0(t,\ \tau_1,\ \tau_2) = e^{-\tau_1}\left[Cos\left(\frac{\tau_2}{2}\right) Cost + Sin\left(\frac{\tau_2}{2}\right) Sint \right]$$

$$= e^{-\tau_1}\ Cos\left(t\ -\ \frac{\tau_2}{2}\right)$$

$$u_1(t,\ \tau_1) = e^{-\tau_1}\ Sint.$$

Finally, the solution to the ε^2 equation, given the initial conditions $u_2(0) = 0$ and

$\frac{\partial u_2}{\partial t}(0) = \frac{\partial u_0}{\partial \tau_2}(0,\ 0,\ 0) - \frac{\partial u_1}{\partial \tau_1}(0,\ 0) = 0.$ is simply $u_2(t) = 0$

The resulting multiple scale solution.

$$u(t,\ \varepsilon) = e^{-\varepsilon t}\left[Cos\left(1-\frac{\varepsilon^2}{2}\right)t + \varepsilon \sin t\right] + o(\varepsilon^2).$$

is compared with the exact solution.

2.2 Two-Variable Expansion

Instead of introducing many slow variables $\tau_n = \varepsilon^n t$ for n = 1, 2, . . ., N, it is often more convenient to consider only two time variables: the slow variable $\tau = \varepsilon t$ and the modified fast variable

$$T = (1+\varepsilon^2 v_2 + \varepsilon^3 v_3 + \ldots + \varepsilon^N v_N)t$$

for some constants v_j.

We can then express

$$\frac{\partial}{\partial t} = (1+\varepsilon^2 v_2 + \varepsilon^3 v_3 + \ldots + \varepsilon^N v_N)\frac{\partial}{\partial T} + \varepsilon \frac{\partial}{\partial \tau}$$

Consider again

$$\begin{cases} u'' + 2\varepsilon u' + u = 0 & t \in [0, \infty), \\ u(0,\varepsilon) = 1, & u'(0,\varepsilon) = 0, \end{cases}$$

and introduce new time scales $\tau = \varepsilon t$ *and* $T = (1+\varepsilon^2 v)t$ and let $u(t,\varepsilon) = u(T\ \ \tau,\ \varepsilon)$,

we find

$$u' = (1+\varepsilon^2 v)\frac{\partial u}{\partial T} + \varepsilon \frac{\partial u}{\partial \tau}$$

$$u'' = \left[\left(1+\varepsilon^2 v\right)\frac{\partial}{\partial T} + \varepsilon \frac{\partial}{\partial \tau}\right]^2 u$$

$$= \frac{\partial^2 u}{\partial T^2} + 2\varepsilon \frac{\partial^2 u}{\partial T \partial \tau} + \varepsilon^2\left(2r\frac{\partial^2 u}{\partial T^2} + \frac{\partial^2 u}{\partial \tau^2}\right) + o(\varepsilon$$

The problem then becomes:

$$\begin{cases} \frac{\partial^2 u}{\partial T^2} + u + 2\varepsilon\left(\frac{\partial^2 u}{\partial T \partial \tau} + \frac{\partial u}{\partial T}\right) + \varepsilon^2\left(2v\frac{\partial^2 u}{\partial T^2} + \frac{\partial^2 u}{\partial \tau^2} + 2\frac{\partial u}{\partial \tau}\right) + o(\varepsilon^2) \\ u(0,0,\varepsilon) = 1 \\ \frac{\partial u}{\partial T}u(0,0,\varepsilon) + \varepsilon\frac{\partial u}{\partial \tau}u(0,0,\varepsilon) + \varepsilon^2 v\frac{\partial u}{\partial T}u(0,0,\varepsilon) = 0 \end{cases}$$

We look for a solution to this partial differential equation of the form

$$u(T,\ \tau,\ \varepsilon) = u_0(T,\ \tau) + \varepsilon u_1(T,\ \tau) + \varepsilon^2 u_2(T,\ \tau) + o(\varepsilon^2).$$

We find

$$\varepsilon^0 : \begin{cases} \frac{\partial^2 u_0}{\partial T^2} + u_0 = 0 \\ u_0(0,\ 0) = 1 \\ \frac{\partial u_0}{\partial T}(0,\ 0) = 0 \end{cases}$$

$$\varepsilon^1 : \begin{cases} \frac{\partial^2 u_1}{\partial T^2} + u_1 = -2\frac{\partial^2 u_0}{\partial T \partial \tau_1} - 2\frac{\partial u_0}{\partial T} \\ u_1(0,\ 0) = 0 \\ \frac{\partial u_1}{\partial T}(0,\ \ 0) = -\frac{\partial u_0}{\partial \tau}(0,\ 0) \end{cases}$$

$$\varepsilon^2 : \begin{cases} \frac{\partial^2 u}{\partial T^2} + u_2 = -2v\frac{\partial^2 u_0}{\partial T^2} - \frac{\partial^2 u_0}{\partial \tau^2} - 2\frac{\partial u_0}{\partial \tau} - 2\frac{\partial^2 u_1}{\partial T \partial \tau} - 2\frac{\partial u_0}{\partial T} \\ u_2 = (0,0) = 0 \\ \frac{\partial u_2}{\partial T}(0,0) = -\frac{\partial u_1}{\partial \tau}(0,0) - v\frac{\partial u_0}{\partial T}(0,0) \end{cases}$$

The solution to the ε^0 problem is

$$u_0(T,\ \tau) = A(\tau)Cos\ T\ +\ B(\tau)Sin\ T.$$

When the initial conditions on u_0 imply that $A(0) = 1$ and $B(0) = 0$, ε^1 problem is then

$$\frac{\partial^2 u_1}{\partial t^2} + u_1 = 2(A' + A)SinT - 2(B' + B)CosT$$

We avoid secular terms by setting

$$A' + A = 0$$

$$B' + B = 0$$

using the initial conditions A(0) = 1 and B(0) = 0, yields A(τ) = $e^{-\tau}$ and B(τ) = 0. We thus find

$$u_0(T,\ \tau) = e^{-\tau_1} CosT$$

$$u_1(T,\ \tau_1) = C(\tau)CosT + D(\tau)\,SinT$$

The initial conditions on $u_1$then implies that C(0) = 0 and D(0) = 1, ε^2 equation thus becomes,

$$\frac{\partial^2 u_2}{\partial t^2} + u_2 = [-2(D' + D) + (1 + 2v)e^{-\tau}]cosT + 2(C' + C)sinT$$

Again, we remove the secular terms by setting

$$C' + C = 0$$

$$D' + D = \frac{1}{2}(1 + 2v)e^{-\tau}$$

With the initial conditions C(0) = 0 and D(0) =

1. We thus find C(τ) = 0 and

$D(\tau) = \left[1 + \tfrac{1}{2}(1+2v)\tau\right]e^{-\tau}$, so that

$$u(T,\ \tau) \approx u_0 + \varepsilon u_1 = e^{-\tau}\left\{CosT + \varepsilon\left[1 + \tfrac{1}{2}(1+2v)\tau\right]\sin T\right\}$$

We however see that these solutions still contains secular term. Fortunately, we still have enough freedom to suppress this secularity, we need only choose v = - ½ , so that D(τ) = $e^{-\tau}$ and

$$u(T,\ \tau) \approx u_0 + \varepsilon u_1 = e^{-\tau}\left(CosT + \varepsilon \sin T\right)$$

Thus that the solution reproduced the exact solution more closely than the approximation obtained previously by the derivative expansion method.

$$u(t,\varepsilon) \approx u\left(\left(1 - \frac{1}{2}\varepsilon^2\right)t, \varepsilon t\right)$$
$$= e^{-zt}\left[Cos\left(1 - \frac{1}{2}\varepsilon^2\right)t + \varepsilon Sin\left(1 - \frac{1}{2}\varepsilon^2\right)t\right]$$

We observe for a relative large value of ε that this solution reproduces the exact solution more closely than the approximate obtained previously by the derivative method.

The perturbation expansion is invalid since it attempts to separate the true dependence of **u** on **t** and **ε** into a series containing products of functions of **t** and functions of **ε**; the exact solution evidently cannot be written in this form. Instead, we see that for small ε there are really two time scales, **t** and **εt**,

3. Numerical Examples

We consider here some selected examples for experimentation with the methods derived in this paper, that is, derivative expansion method and Two- variable expansion methods of Asymptotic Multiple–Scale Analysis. The experiments were carried out by the Mathematical software (MATLAB 2009b) and the results were presented below.

Example 1: Consider the problem (see [16])

$$\begin{cases} \varepsilon u'' + 2u' + e^{u} = 0 & x \in [0, 1] \\ u(0,\ \varepsilon) = u(1,\ \varepsilon) = 0 \end{cases}$$

The composite solution is

$$u(x,\varepsilon) = \log\frac{2}{1+x} - e^{-2x/\varepsilon}\log 2$$

(take ε = 0.05)

Example2: (see [16])

$$\begin{cases} \varepsilon u'' - x^2u' - u = 0 & x \in [0,1] \\ u(0,\varepsilon) = u(1,\varepsilon) = 0 \end{cases}$$

The composite solution is

$$u(x,\varepsilon) = e^{\frac{x}{\sqrt{z}}} + e^{\frac{-(1-x)}{\sqrt{z}}}$$

Take $\varepsilon = 0.02$

Example 3: (see [16])

$$\begin{cases} \varepsilon u'' + (1+\varepsilon)u' + u = 0 & x \in [0,1] \\ u(0,\varepsilon) = \alpha,\ u(1,\varepsilon) = \beta \end{cases}$$

The exact solution to P is

$$u(x,\varepsilon) = \frac{\left(\beta - \alpha e^{-1/\varepsilon}\right)e^{1-x} + (\alpha - \beta e)e^{-x/\varepsilon}}{1 - e^{1-1/\varepsilon}}$$

Table of Result of the findings

Table 1 (for Example 1)

x\ε	0.05	0.07	0.10	0.15
0	0	0	0	0
0.1	2.541237538E-1	2.42348369E-1	2.188973308E-1	1.802866652E-1
0.2	2.217477653E-1	2.208558004E-1	2.163351929E-1	2.009321466E-1
0.3	1.87-0847938E-1	1.870296157E-1	1.863404646E-1	1.815730867E-1
0.4	1.549019261E-1	1.548986847E-1	1.548009757E-1	1.534486072E-1
0.5	1.24938736E-1	1.249385485E-1	1.249250699E-1	1.245556357E-1
0.6	9.6910013E-2	9.69100022E-2	9.690816342E-2	9.680902869E-2
0.7	7.058107429E-2	7.058107367E-2	7.058082397E-2	7.055445511E-2
0.8	4.575749056E-2	4.575749053E-2	4.575745668E-2	4.575047382E-2
0.9	2.227639471W-2	2.227639471E-2	2.227639013E-2	2.227454512E-2
1.0	-1.278882063E-18	-1.175427924E-13	-6.20469066E-10	-4.875472154E-7

Table 2 (for Example 2)

x\ε	0.02	0.05	0.07	0.10
0	1	1.00000002	1.000000025	1
0.1	2.028114382E0	6.394073344E-1	6.852574527E-1	7.290168239E-1
0.2	2.431167344E-1	4.088418323E-1	4.695850831E-1	5.316210718E-1
0.3	1.198732501E-1	2.614172195E-1	3.218233977E-1	3.881624635E-1
0.4	5.910574656E-2	1.671576961E-1	2.206893739E-1	2.847431506E-1
0.5	2.914319313E-2	1.069233256E-1	1.518891372E-1	2.124786081E-1
0.6	1.436959815E-2	6.867399055E-2	1.068395856E-1	1.682786518E-1
0.7	7.085503843E-3	4.617490711E-2	8.47188134E-2	1.590941208E-1
0.8	3.538889206E-3	4.625526017E-2	1.060528393E-1	2.150084739E-1
0.9	8.460477185E-3	1.532000944E-1	2.729682778E-1	4.259527051E-1
1.0	1.000849326E0	1.01142289E0	1.022830801E0	1.367879441E0

The value of the solution is still very closed, despite the variant in the value of ε of the approximate solution of the two examples above

4. Conclusion

Based on the result of the findings, the method shows the reliability and stability of the two methods especially when the value of ε is at variants. The table of the results confirm these analogies. Also, it gives better accuracy with low cost of implementation within a very short time.

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